

# ON A FACTORIZATION OF GRADED HOPF ALGEBRAS USING LYNDON WORDS

M. GRAÑA AND I. HECKENBERGER

**ABSTRACT.** We find a generalization of the restricted PBW basis for pointed Hopf algebras over abelian groups constructed by Kharchenko. We obtain a factorization of the Hilbert series for a wide class of graded Hopf algebras. These factors are parametrized by Lyndon words, and they are the Hilbert series of certain graded Hopf algebras.

**Key words:** graded Hopf algebra, Nichols algebra, Lyndon word

**MSC2000:** 16W30; 16W50

## 1. INTRODUCTION

Hopf algebras [Swe69] are far from being classified. Up to now there are two main directions of study: semisimple and pointed Hopf algebras. This paper is mainly a contribution to the latter, although all considerations are performed in a more general context. Specifically, we work with Hopf algebras  $H$  generated by a Hopf subalgebra  $H_0$  and a vector space  $V$  satisfying properties (3.2) and (3.3) below. This includes in particular pointed Hopf algebras generated by group-like and skew-primitive elements. In Theorem 3.12 we prove a factorization result about the Hilbert series of  $\text{gr } H$ . Moreover, with Theorem 3.19 we show that to each factor one can associate in a natural way a graded Hopf algebra which projects onto a Nichols algebra. These are the main results of the present paper.

Kharchenko [Kha99] proved that when  $H_0$  is the group algebra of an abelian group and it acts on  $V$  by characters,  $H$  admits a restricted PBW basis. The PBW generators in this basis are labelled by Lyndon words on an alphabet given by a set of skew-primitive elements. Examples, where  $H_0$  is the group algebra of a nonabelian group [MS00] [Gra00] [AG03], indicate that in general one can not expect Kharchenko's result to hold in its strong form. Nevertheless,

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by extending his ideas we were able to construct a basis of  $\text{gr } H$  using ordered products of subquotients of it. In the particular case of Kharchenko's setting one recovers the PBW basis. This is possible because graded braided Hopf algebras generated by one primitive element are easy to classify. The difficulty in the more general setting arises from the fact that the structure of graded braided Hopf algebras generated by an irreducible Yetter–Drinfel'd module over  $H_0$  is not known.

A generalization of Kharchenko's PBW theorem in a different direction was done by Ufer [Ufe04]. Instead of Hopf algebras of diagonal type he considers Hopf algebras with triangular braidings. On the one hand Ufer is able to give a restricted PBW basis. On the other hand some information about the relations of the Hopf algebra is lost. Although we believe that it is possible to obtain a generalization of Ufer's approach to our context, we stick to a simpler setting for two reasons. First, valuable additional information can be obtained in our starting context. Second, the proofs in the triangular case would be even more technical, obscuring the essential arguments.

The proof of the main results of the present paper was possible due to taking advantage both from the lexicographic and the inverse lexicographic order on the set of monotonic super-words built from an alphabet of Lyndon words. This leads in a natural way to the construction of subquotients of a graded Hopf algebra. In this way Kharchenko's PBW theorem becomes more transparent. Note that Ufer's more technical proof stems from the fact that in his setting the inverse lexicographic order on the set of monotonic super-words can not be used.

Kharchenko's PBW theorem turned out to be essential in the construction of the Weyl groupoid [Hec06] corresponding to a Nichols algebra of diagonal type. This groupoid played the crucial role in the classification of such Nichols algebras [Hec04]. In turn, the knowledge of these Nichols algebras is important for example for the lifting method of Andruskiewitsch and Schneider [AS98] to classify pointed Hopf algebras. We consider the factorization theorem in this paper to be an important step towards the generalization of the Weyl groupoid to a wider class of Nichols algebras.

In this paper  $\mathbf{k}$  is an arbitrary field, and all algebras have  $\mathbf{k}$  as their base field. The symbol  $\otimes$  refers to tensor product over  $\mathbf{k}$ . We will write  $m$ ,  $\Delta$ ,  $\varepsilon$  and  $S$  for the product, coproduct, counit, and the antipode of a Hopf algebra.

We conclude the introduction with two general results about Hopf algebras, for which we did not find references in the literature.

**Proposition 1.1.** *Let  $A = (A, m, \Delta)$  be a bialgebra,  $B \subseteq A$  a subalgebra and  $I \subseteq B \cap \ker(\varepsilon)$  an ideal of  $B$ . Suppose furthermore that*

$$(1.2) \quad \Delta(B) \subseteq B \otimes B + I \otimes A, \quad \Delta(I) \subseteq B \otimes I + I \otimes A.$$

*Then the bialgebra structure on  $A$  induces a bialgebra structure on  $B/I$ .*

*Proof.* Notice first that  $B/I$  is an algebra. Let us take  $\bar{\Delta} : B \rightarrow A/I \otimes A/I$  as  $\bar{\Delta} = (\pi \otimes \pi)\Delta i$ , for  $\pi : A \rightarrow A/I$  the projection and  $i : B \rightarrow A$  the inclusion. By using the first formula in (1.2), one gets that  $\bar{\Delta}(B) \subseteq B/I \otimes B/I$ . By using the second formula in (1.2), one gets that  $\bar{\Delta}(I) = 0$ . Then  $\bar{\Delta}$  induces a map  $\tilde{\Delta} : B/I \rightarrow B/I \otimes B/I$ . Also,  $\varepsilon$  induces a map  $\tilde{\varepsilon} : B/I \rightarrow \mathbf{k}$ . It is clear that  $\tilde{\Delta}$  is coassociative and  $\tilde{\varepsilon}$  is a counit for  $\tilde{\Delta}$ , whence  $B/I$  is a coalgebra. It is immediate that  $\tilde{\varepsilon}$  is an algebra map. We must prove then that  $\tilde{\Delta}$  is an algebra map. For  $a, b \in B$  we compute

$$\begin{aligned} \tilde{\Delta}(\pi(a)\pi(b)) &= \tilde{\Delta}(\pi(ab)) = (\pi \otimes \pi)\Delta(ab) = (\pi \otimes \pi)(a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)}) \\ &= \pi(a_{(1)}b_{(1)}) \otimes \pi(a_{(2)}b_{(2)}) = \pi(a_{(1)})\pi(b_{(1)}) \otimes \pi(a_{(2)}b_{(2)}) \\ &= \pi(a_{(1)})\pi(b_{(1)}) \otimes \pi(a_{(2)})\pi(b_{(2)}). \end{aligned}$$

In the first equality above we used that  $\pi|_B : B \rightarrow B/I$  is an algebra map, and the fifth one is obtained from  $a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)} \in B \otimes A$ . The last equality holds by  $\pi(a_{(1)})\pi(b_{(1)}) \otimes a_{(2)}b_{(2)} \in B/I \otimes B$ .  $\square$

For the notion of braided Hopf algebras one may consult for example [Tak00].

**Proposition 1.3.** *Let  $\pi : R \rightarrow T$  be a surjective map of braided graded Hopf algebras which is an isomorphism in degree 0, and assume that  $R, T$  are finite dimensional in each degree. Then the quotient  $\eta(R, t)/\eta(T, t)$  of Hilbert series is again a series with nonnegative integer coefficients.*

*Proof.* Since  $\ker \pi$  is homogeneous, the spaces  $(\ker \pi)^i/(\ker \pi)^{i+1}$  inherit the grading of  $R$ . Set

$$R' = \bigoplus_{i=0}^{\infty} (\ker \pi)^i/(\ker \pi)^{i+1}, \text{ where } (\ker \pi)^0/(\ker \pi)^1 = R/(\ker \pi) \simeq T.$$

Then  $R$  and  $R'$  are isomorphic as graded vector spaces. Moreover,  $R'$  is a braided Hopf algebra, and  $T \subset R$  is a braided Hopf subalgebra with a projection  $\pi' : R' \rightarrow T$ . Therefore the braided version of the Radford-Majid theorem gives that  $R' \simeq R'^{\text{co } T} \otimes T$  as graded vector spaces. This implies that  $\eta(R, t) = \eta(R', t) = \eta(R'^{\text{co } T}, t)\eta(T, t)$ .  $\square$

## 2. LYNDON WORDS

Let  $A = \{\mathbf{1}, \mathbf{2}, \dots, \mathbf{d}\}$  be a totally ordered set by  $\mathbf{1} < \mathbf{2} < \dots < \mathbf{d}$ . We think of  $A$  as an *alphabet* and  $\mathbf{1}, \dots, \mathbf{d}$  as the *letters* of  $A$ . Let  $\mathbb{A}$  be the set of non-empty words in this alphabet, and let  $\emptyset$  denote the empty word. For a word  $u = a_1 a_2 \dots a_r$  with  $a_i \in A \ \forall i$ , we say that  $r$  is the *length* of  $u$  and we write  $r = |u|$ . We consider on  $\mathbb{A}$  the lexicographic order  $<$ . This means that  $u < v$  if and only if  $v = uu'$  for some  $u' \in \mathbb{A}$  or  $u = wi\mathbf{u}'$  and  $v = w\mathbf{j}v'$ , where  $\mathbf{i} < \mathbf{j}$  and  $w, u', v' \in \{\emptyset\} \cup \mathbb{A}$ .

A word  $u \in \mathbb{A}$  is called a *Lyndon word* if  $u = u_1 u_2$  with  $u_1, u_2 \in \mathbb{A}$  implies that  $u < u_2$ . For example: letters are Lyndon words,  $\mathbf{ij}$  is a Lyndon word for  $\mathbf{i} < \mathbf{j}$ ,  $\mathbf{12122}$  is a Lyndon word, and  $\mathbf{1212}$  is not a Lyndon word. We write  $L = \{u \in \mathbb{A} \mid u \text{ is a Lyndon word}\}$ .

**Proposition 2.1.** [Lot83, Prop. 5.1.3] *A word  $u$  is a Lyndon word if and only if  $u \in A$  or  $u = vw$  with  $v, w \in L$  and  $v < w$ . More precisely, if  $w$  is the proper right factor of maximal length of  $u = vw \in L$  that belongs to  $L$ , then also  $v \in L$  and  $v < vw < w$ .*  $\square$

If  $u \in L$ ,  $|u| \geq 2$ , then the decomposition  $u = vw$  in Prop. 2.1 with  $w$  of maximal length is called the *Shirshov decomposition* of  $u$ . We write  $\text{III}(u) = (v, w)$ .

**Lemma 2.2.** *Let  $u, v \in L$ ,  $u < v$ ,  $|u| \geq 2$ , and let  $\text{III}(u) = (u_1, u_2)$ . Then exactly one of the following possibilities occurs:*

- (1)  $\text{III}(uv) = (u, v)$  and  $u_2 \geq v$ .
- (2)  $\text{III}(uv) = (u'_1, u''_1 u_2 v)$  for some words  $u'_1, u''_1$  such that  $u_1 = u'_1 u''_1$  and  $u_2 < v$  (here  $u''_1$  may be empty).

*Proof.* This is equivalent to [Lot83, Prop. 5.1.4]  $\square$

**Lemma 2.3.** [Kha99, Lemma 4] *Let  $u, v \in L$ ,  $u = u_1 u_2$  for  $u_1, u_2 \in \mathbb{A}$  and suppose that  $u_2 < v$ . Then  $uv < u_1 v$ .*  $\square$

We take on  $L$  the lexicographic order. Thus,  $L$  is a new alphabet containing the original alphabet  $A$ , and following Kharchenko [Kha99] we say that the elements of  $L$  are *super-letters*. Words in super-letters are called *super-words*. The *length*  $|w|$  of a super-word  $w$  is the sum of the lengths of its super-letters. A *monotonic super-word* is a non-increasing word on the set of super-letters, i.e., a (possibly empty) word  $v_1 \dots v_n$  such that  $v_i \in L$  and  $v_1 \geq v_2 \geq \dots \geq v_n$ . Let  $M$  denote the set of monotonic super-words. In what follows the notation  $v_1 \dots v_n \in M$  will mean that  $v_1, \dots, v_n \in L$  and  $v_1 \geq \dots \geq v_n$ . Sometimes we also write  $v_1^{m_1} \dots v_n^{m_n} \in M$ , in which case we mean that  $v_1, \dots, v_n \in L$ ,  $v_1 > \dots > v_n$ , and  $m_1, \dots, m_n \geq 1$ . Monotonic super-words are

lexicographically ordered on the alphabet of super-letters. Notice that the empty super-word is the smallest super-word. For a super-letter  $u$ , we shall write

$$(2.4) \quad \begin{aligned} L_{>u} &= \{v \in L \mid v > u\}, \quad L_{\geq u} = \{v \in L \mid v \geq u\}, \\ M_{>u} &= \{v_1 \cdots v_r \in M \mid r \geq 1, v_1 \geq \cdots \geq v_r > u\}, \\ M_{\geq u} &= \{v_1 \cdots v_r \in M \mid r \geq 1, v_1 \geq \cdots \geq v_r \geq u\}. \end{aligned}$$

**Theorem 2.5.** (Lyndon, see [Lot83, Thm. 5.1.5 and Prop. 5.1.6]) *A word in  $\mathbb{A}$  can be written in a unique way as a monotonic super-word. Moreover, if  $u = v_1 \cdots v_n \in M$  then  $v_n$  is the smallest right factor of  $u$  (smallest with respect to lexicographic order in  $\mathbb{A}$ ).*  $\square$

As an example, the word **1231233122123** is decomposed as a monotonic super-word as

$$1231233122123 = (1231233)(122123),$$

and in turn,  $\text{III}(1231233) = (123, 1233)$ .

**Lemma 2.6.** *Let  $w$  be a super-word. Then the decomposition of  $w$  as a monotonic super-word is  $\geq w$  with respect to the lexicographic order on super-words.*

*Proof.* We proceed by induction on the number of super-letters of  $w$ . If  $w$  is a super-letter then we are done. Otherwise  $w = w_1 \cdots w_n$ , where  $n \geq 2$  and  $w_i \in L \forall i$ . Again, if  $w_1 \geq w_2 \geq \cdots \geq w_n$  then we are done. On the other hand, if  $w_i < w_{i+1}$  for some  $i$ , then  $w'_i := w_i w_{i+1} \in L$  by Proposition 2.1, and one has  $w = w_1 \cdots w_{i-1} w'_i w_{i+2} \cdots w_n$  with  $w'_i > w_i$ . Hence the claim follows from the induction hypothesis.  $\square$

**Lemma 2.7.** [Kha99, Lemma 5] *Let  $w = w_1 \cdots w_m$  and  $v = v_1 \cdots v_n$  be monotonic super-words. Then  $w < v$  (with respect to the lexicographic order on  $M$ ) if and only if  $w_1 \cdots w_m < v_1 \cdots v_n$  with respect to the lexicographic order on  $\mathbb{A}$ .*  $\square$

The following technical lemma will be needed in the proof of Theorem 3.12.

**Lemma 2.8.** *Let  $w = w_1 \cdots w_m$  and  $v = v_1 \cdots v_n$  be nonempty monotonic super-words with  $w \geq v$  and assume that  $v_1 = \cdots = v_r > v_{r+1}$ , where  $r \leq n$ . For all  $i \leq m$  let  $(w'_i, w''_i) \in M \times M$  such that either  $w'_i \geq w_i$  or  $w'_i = \emptyset$ ,  $w''_i = w_i$ . Then the pair  $(w'_1 \cdots w'_m, w''_1 \cdots w''_m)$  satisfies one of the following relations.*

- (1)  $w'_1 \cdots w'_m > v_{r+1} \cdots v_n$ ,
- (2)  $w''_1 \cdots w''_m > v_1^r$ ,
- (3)  $(w'_1 \cdots w'_m, w''_1 \cdots w''_m) = (v_{r+1} \cdots v_n, v_1^r)$ .

Moreover, if  $w > v$  then (3) can not hold.

*Proof.* Assume first that  $w_1 > v_1$ . Then either  $w'_1 > v_1 > v_{r+1} \cdots v_n$  or  $w'_1 = \emptyset$ ,  $w''_1 = w_1 > v_1$ . In the first case we have relation (1) and in the second one relation (2) is fulfilled. On the other hand, if  $w_1 \leq v_1$  then because of  $w \geq v$  and  $w$  is monotonic, one has  $m \geq r$  and  $w_i = v_1$  for all  $i \leq r$ . Suppose first that there exists  $i \leq r$  such that  $w'_i \neq \emptyset$  and  $w'_j = \emptyset$  for all  $j < i$ . In this case  $w'_i \geq v_1 > v_{r+1} \cdots v_n$  and hence (1) holds. It remains to consider the case  $w'_i = \emptyset$  for all  $i \leq r$ . Then one has  $w''_1 \cdots w''_r = v_1^r$ . Therefore, if  $w''_i \neq \emptyset$  for some  $i > r$  then again relation (2) holds. Otherwise  $w'_i \geq w_i$  for all  $i > r$ . Then one has either  $w'_1 \cdots w'_m = w'_{r+1} \cdots w'_m > v_{r+1} \cdots v_n$ , in which case (1) holds, or  $m = n$ ,  $w = v$ , and  $w'_i = w_i$  for all  $i > r$ . The latter relations imply (3).  $\square$

Let  $H_0$  be a Hopf algebra and let  $V = \bigoplus_{i=1}^d V_i$  be a direct sum of Yetter–Drinfel’d modules over  $H_0$ . Let  $TV$  be the tensor algebra of  $V$ . For simplicity, we will omit the  $\otimes$  symbol in products of elements of  $TV$ . Let  $\cdot$  and  $\delta$  denote the left action and the left coaction of  $H_0$  on  $TV$ , respectively. We will use Sweedler notation  $\delta(x) = x_{(-1)} \otimes x_{(0)}$  for  $x \in TV$ . The algebra  $TV$  has a braiding  $c : TV \otimes TV \rightarrow TV \otimes TV$ . Note that one has

$$c(x \otimes y) = x_{(-1)} \cdot y \otimes x_{(0)}, \quad c^{-1}(x \otimes y) = y_{(0)} \otimes S^{-1}(y_{(-1)}) \cdot x$$

for all  $x, y \in TV$ . We define

$$[x, y] = xy - mc^{-1}(x \otimes y), \quad \llbracket x, y \rrbracket = xy - mc(x \otimes y).$$

where  $m$  is the multiplication in  $TV$ .

**Definition 2.9.** Let  $a_1, \dots, a_m \in A$  and  $u = a_1 \dots a_m \in \mathbb{A}$ . We write  $V^u = V_{a_1} V_{a_2} \cdots V_{a_m} \simeq V_{a_1} \otimes V_{a_2} \otimes \dots \otimes V_{a_m}$ . The elements in  $V^u$  will be called  $u$ -vectors. If  $x_u \in V^u$  and  $u = vw$ , then we write  $x_u = x_v \otimes x_w \in V^v \otimes V^w$  (which is in general a sum of tensors) using the canonical isomorphism  $V^u \simeq V^v \otimes V^w$ .

We shall inductively define bracket operations  $[ ] : \bigoplus_{n \geq 0} V^{\otimes n} \rightarrow TV$  and  $\llbracket \rrbracket : \bigoplus_{n \geq 0} V^{\otimes n} \rightarrow TV$  as follows. Let  $x_u$  be a  $u$ -vector.

- (1) If  $u$  has length 0 or 1, then  $[x_u] = \llbracket x_u \rrbracket = x_u$ .
- (2) If the word  $u$  is a Lyndon word and  $\text{III}(u) = (v, w)$ , then  $[x_u] = [[x_v], [x_w]]$  and  $\llbracket x_u \rrbracket = [[\llbracket x_v \rrbracket, \llbracket x_w \rrbracket]]$  (see Def. 2.9).
- (3) If the word  $u$  is decomposed as a monotonic super-word by  $u = v_1 \cdots v_r$ , then  $[x] = [x_{v_1}] \cdot [x_{v_2}] \cdots [x_{v_r}]$  and  $\llbracket x \rrbracket = \llbracket x_{v_1} \rrbracket \cdot \llbracket x_{v_2} \rrbracket \cdots \llbracket x_{v_r} \rrbracket$ .

*Remark 2.10.* Recall that the braided antipode  $S_{TV}$  of the braided Hopf algebra  $TV$  satisfies  $S_{TV}(x) = x_{(-1)}S(x_{(0)})$ , where  $S$  is the antipode of  $TV \# H_0$ . Moreover,  $S_{TV}(xy) = \text{mc}(S_{TV}(x) \otimes S_{TV}(y)) = (x_{(-1)}S_{TV}(y))S_{TV}(x_{(0)})$ . With these formulas it is easy to see that for any  $u \in L$  and any  $u$ -vector  $x$  one has  $S_{TV}([x]) = (-1)^{|u|-1}[[x]]$ . Therefore most of the following considerations can be performed without difficulties with  $[[\cdot]]$ 's instead of  $[\cdot]$ 's. Even if  $[[\cdot]]$ 's seem to be more natural, we will follow the tradition of Kharchenko [Kha99] with  $[\cdot]$ 's.

**Definition 2.11.** If  $u$  is a Lyndon word and  $x$  is a  $u$ -vector, then  $[x]$  will be called a  $u$ - $[\cdot]$ -letter. If  $u$  is a monotonic super-word and  $x$  is a  $u$ -vector then  $[x]$  will be called a  $u$ - $[\cdot]$ -word. Let  $V^{[u]} = [V^u]$  denote the space of  $u$ - $[\cdot]$ -words. A  $[\cdot]$ -letter ( $[\cdot]$ -word) is a  $u$ - $[\cdot]$ -letter ( $u$ - $[\cdot]$ -word) for some super-letter (monotonic super-word)  $u$ .

**Lemma 2.12.** Let  $u$  be a monotonic super-word. For  $x, y \in TV$ , let  $c^{-1}(x \otimes y) =: \sum {}^x y \otimes {}^x y$ . Then

- (1) if  $x$  is a  $u$ -vector and  $h \in H_0$  then one has  $[h.x] = h.[x]$  and  $[x]_{(-1)} \otimes [x]_{(0)} = x_{(-1)} \otimes [x]_{(0)}$ ;
- (2) if  $x$  is a  $u$ -vector and  $y$  is a  $v$ -vector for a monotonic super-word  $v$ , then  $[{}^x y] \otimes [x]^y = {}^x y \otimes [x]^y$  and  ${}^x [y] \otimes x^{[y]} = [{}^x y] \otimes x^{[y]}$ ;
- (3) for all  $x, y, z \in TV$ ,

$$[x, yz] = [x, y] z + \sum {}^x y [x^y, z];$$

- (4) for all  $x, y, z \in TV$ ,

$$[[x, y], z] - [x, [y, z]] = \sum [x, {}^y z] y^z - \sum {}^x y [x^y, z].$$

*Proof.* (1) follows from the fact that the braiding is a map of Yetter–Drinfel'd modules. (2) follows from (1) and the braid relation. (3) and (4) are straightforward calculations using the definition of  $[\cdot]$  and the braid relation for  $c^{-1}$ .  $\square$

We prove now a variant of [Kha99, Lemma 6].

**Lemma 2.14.** Let  $X, Y$  be a  $u$ - $[\cdot]$ -letter and a  $v$ - $[\cdot]$ -letter respectively, where  $u, v \in L$  and  $u < v$ . Then  $[X, Y]$  is a homogeneous linear combination of products of  $[\cdot]$ -letters corresponding to super-letters in  $L_{\geq uv}$ .

*Proof.* Let  $z = uv$ . If  $\text{III}(z) = (u, v)$ , then  $[X, Y]$  is a  $z$ - $[\cdot]$ -letter and we are done. We proceed by induction on  $|uv|$ , since we already know the lemma for the case  $|u| = |v| = 1$ .

Let  $m = |uv|$  and suppose that the lemma holds for  $u', v' \in L$ ,  $u' < v'$ ,  $X \in V^{[u']}$ ,  $Y \in V^{[v']}$ , where either  $|u'v'| < m$  or  $|u'v'| = m$ ,

$u' < u$  (notice that there are only finitely many Lyndon words of a given length). As noted above, we may suppose that  $\text{III}(uv) \neq (u, v)$ . Then, if  $\text{III}(u) = (u_1, u_2)$ , we must have  $u_2 < v$  because of Lemma 2.2. Let  $X = [x]$  for  $x \in V^u$ ,  $Y = [y]$  for  $y \in V^v$ , and let  $x = x_u = x_{u_1} \otimes x_{u_2}$ ,  $X_{u_1} \otimes X_{u_2} = [x_{u_1}] \otimes [x_{u_2}]$ . Thanks to Lemma 2.12(4),(2), we have

$$\begin{aligned}
 [X, Y] &= [[X_{u_1}, X_{u_2}], Y] \\
 &= [X_{u_1}, [X_{u_2}, Y]] \\
 (2.15) \quad &\quad + \sum [X_{u_1}, {}^{X_{u_2}}Y](X_{u_2})^Y - \sum {}^{X_{u_1}}X_{u_2}[X_{u_1}^{X_{u_2}}, Y] \\
 &= [X_{u_1}, [X_{u_2}, Y]] \\
 &\quad + \sum [X_{u_1}, [{}^{x_{u_2}}y]] \cdot [x_{u_2}^y] - \sum [{}^{x_{u_1}}x_{u_2}] \cdot [[x_{u_1}^{x_{u_2}}], Y].
 \end{aligned}$$

We start by considering the first summand on the right hand side. By the induction hypothesis,  $[X_{u_2}, Y]$  is a homogeneous linear combination of products of  $[]$ -letters corresponding to super-letters in  $L_{\geq u_2v}$ , and the degree of these products is  $|u_2v|$ . By Lemma 2.12(3), and since  $u_1 < u_2 < u_2v$ ,  $[X_{u_1}, [X_{u_2}, Y]]$  is a combination of products of  $[]$ -letters corresponding to super-letters in  $L_{\geq u_2v} \subset L_{>w}$  on one hand, and a bracket  $[h.X_{u_1}, X']$  on the other hand, where  $h \in H_0$  and  $X'$  is a  $w$ - $[]$ -letter with  $w \geq u_2v$ . By induction hypothesis again, the latter is a linear combination of products of  $[]$ -letters corresponding to super-letters in  $L_{\geq u_1u_2v} = L_{\geq uv}$ .

We continue with the second and third summands in (2.15). Concerning the  $u_2$ - $[]$ -letters appearing there, notice that  $u_2 > uv$ . Indeed,  $u_2 > u_1u_2$  and, since  $u_1u_2$  is not the beginning of  $u_2$ , then we still have  $u_2 > u_1u_2v = uv$ . On the other factors, which are brackets between  $u_1$ - $[]$ -letters and  $v$ - $[]$ -letters, we can apply the induction hypothesis since  $|u_1v| < |uv|$  and  $u_1 < u_1u_2 < u_2 < v$ . These factors are then linear combinations of products of  $[]$ -letters corresponding to super-letters in  $L_{\geq u_1v}$ , and  $u_1v > uv$  by Lemma 2.3.  $\square$

**Lemma 2.16.** *Let  $u \in L$ . Any product of  $[]$ -letters corresponding to super-letters in  $L_{\geq u}$  is a linear combination of (monotonic)  $[]$ -words corresponding to super-words in  $M_{\geq u}$ .*

*Proof.* Let  $t = v_1 \cdot v_2 \cdots v_n$  be a super-word, where  $v_i \in L_{\geq u} \forall i$ . Let  $x = x_t = x_{v_1} \cdots x_{v_n}$  be a  $t$ -vector, where here  $t$  is considered as a word on  $A$ . We call  $x_{[t]} := [x_{v_1}] \cdots [x_{v_n}]$  a  $t$ - $[]$ -vector (notice that the term “ $t$ - $[]$ -word” is reserved for when  $t$  is monotonic). We proceed by induction. We take on the set of super-words the lexicographic order  $\prec$  given by the order  $<$  on  $L$ . Suppose that the result is true for  $w$ - $[]$ -vectors when  $|w| < |t|$  or when  $|w| = |t|$  and  $w \succ t$ . If  $v_1 \geq v_2 \geq \cdots \geq v_n$ ,

there is nothing to prove. Otherwise, let  $i$  be such that  $1 \leq i < n$  and  $v_i < v_{i+1}$ . Let  $t' = v_1 \dots v_{i+1} v_i \dots v_n$  and  $t'' = v_1 \dots (v_i v_{i+1}) \dots v_n$ , where the factor  $(v_i v_{i+1})$  stands for the Lyndon word  $v_i v_{i+1}$ . Let

$$\begin{aligned} x_{[t']} &:= [x_{v_1}] \otimes \dots \otimes c^{-1}([x_{v_i}] \otimes [x_{v_{i+1}}]) \otimes \dots \otimes [x_{v_n}], \\ x_{[t'']} &:= [x_{v_1}] \otimes \dots \otimes [[x_{v_i}], [x_{v_{i+1}}]] \otimes \dots \otimes [x_{v_n}]. \end{aligned}$$

Notice that  $x_{[t']}$  is a  $t'$ -[]-vector where  $t'$  is a super-word in the same super-letters as  $t$ , and  $t' \succ t$ . Also, by Lemma 2.14, and since  $v_i v_{i+1} > v_i$ ,  $x_{[t'']}$  is a linear combination of  $w$ -[]-vectors, where  $w$  runs on super-words  $\succ t$  having only super-letters in  $L_{\geq u}$ . The last thing to notice is that  $x_{[t]} = x_{[t']} + x_{[t'']}$ . Therefore, the induction hypothesis implies the claim.  $\square$

**Definition 2.17.** Let  $x \in TV \setminus \{0\}$ ,  $x = \sum_n x_n$ , where  $x_n \in V^{\otimes n}$ . The greatest  $n$  such that  $x_n \neq 0$  will be called the *degree* of  $x$ . Let  $m$  be the degree of  $x$  and let  $x_m = \sum_{u \in \mathbb{A}_m} x_u$ , where  $\mathbb{A}_m$  is the set of words of length  $m$  and  $x_u$  is a  $u$ -vector for all  $u$ . Let  $v$  be the least  $u$  (with respect to the order in  $\mathbb{A}_m$ ) such that  $x_u \neq 0$ . Then  $x_v$  will be called the *leading vector* of  $x$ .

**Lemma 2.18.** Let  $x \in TV$  be a nonzero  $u$ -vector for a monotonic super-word  $u$ . Then  $x$  is the leading vector of  $[x]$ .

*Proof.* We prove first the lemma for Lyndon words  $u$  by induction on  $|u|$ . If  $|u|$  is 0 or 1, the result is clear. If  $\text{III}(u) = (v, w)$ , then  $[x] = [[x_v], [x_w]] = (m - mc^{-1})([x_v] \otimes [x_w])$ . By the induction hypothesis,  $[x_v][x_w]$  is a sum of products of  $v'$ -vectors and  $w'$ -vectors, where  $v'$  runs on words  $\geq v$  and  $w'$  runs on words  $\geq w$ . For such  $v', w'$  we have  $u = vw \leq v'w'$ , whence  $[x_v][x_w]$  is a sum of  $u'$ -vectors with  $u' \geq u$ . Furthermore, the equality holds if and only if  $v' = v$  and  $w' = w$ . Thus, by using induction hypothesis, the leading vector of  $[x_v][x_w]$  is  $x_v x_w$ . For the term  $mc^{-1}([x_v] \otimes [x_w])$  we reason similarly: as  $u$  is a Lyndon word,  $u = vw < w < wv$ , from where  $mc^{-1}([x_v] \otimes [x_w])$  is a sum of  $u'$ -vectors where  $u'$  runs on words  $\geq wv > u$ . Therefore, such  $u'$ -vectors do not contribute to the leading vector of  $[x_v x_w]$ .

If  $u$  is not a super-letter, then by Theorem 2.5 we have  $u = v_1 \dots v_n \in M$ . By definition  $[x] = [x_{v_1}] \dots [x_{v_n}]$ , which, by the previous step, is a sum of  $v'_1 \dots v'_n$ -vectors with  $v'_i \geq v_i \ \forall i$ , and its leading vector is  $x_{v_1} \dots x_{v_n} = x$ .  $\square$

Recall that  $M$  is the set of monotonic super-words.

**Corollary 2.19.** One has  $TV = \bigoplus_{u \in M} V^{[u]}$ .

*Proof.* Since letters in  $A$  are also super-letters, the spaces  $V^{[u]}$  generate  $TV$  thanks to Lemma 2.16. The linear independence of these spaces follows immediately from Lemma 2.18.  $\square$

**Corollary 2.20.** *Let  $u \in \mathbb{A}$  and  $x$  be a  $u$ -vector. Then  $x - [x]$  is a linear combination of  $w$ -[]-words with  $w > u$ .*

*Proof.* This assertion can be proven with the help of bases of  $\bigoplus_{|v|=|u|} V^v$  and  $\bigoplus_{|v|=|u|} V^{[v]}$  which are obtained from each other using triangular matrices. Alternatively, Lemmas 2.7 and 2.18 imply that  $x - [x]$  is a linear combination of  $w$ -vectors with  $w > u$ , and proceed by induction.

$\square$

Corollary 2.20 allows us to give a description of products of []-letters, which is different from the one in Lemma 2.16.

**Corollary 2.21.** *Let  $n \in \mathbb{N}$ ,  $v_1, \dots, v_n \in L$  and let  $X_i$  be a  $v_i$ -[]-letter  $\forall i$ . Then  $X_1 \cdots X_n$  is a linear combination of  $w$ -[]-words, where  $w$  runs over monotonic super-words  $\geq v'_1 \cdots v'_{n'}$ , and  $v'_1 \cdots v'_{n'}$  is the decomposition of  $v_1 \cdots v_n$  as a monotonic super-word.*  $\square$

**Definition 2.22.** Let  $K$  be a totally ordered set and let there be vector spaces  $W_k$  for each  $k \in K$ . We define  $\otimes_{k \in K}^> W_k$  to be the direct sum of vector spaces  $W_{k_1} \otimes \dots \otimes W_{k_r}$ , where  $k_1 > k_2 > \dots > k_r$ .

For a vector space  $W$ , we write  $T^+W = \bigoplus_{n \geq 1} W^{\otimes n}$ . Notice that for each  $u \in L$ ,  $T^+V^{[u]}$  is a non-unital subalgebra of  $T^+V$  in the category  ${}^{H_0}_{H_0}\mathcal{YD}$ . Further, the  $\mathbb{Z}$ -grading of  $TV$  induces a  $\mathbb{Z}$ -grading on its subalgebras  $TV^{[u]}$ . In particular, the Hilbert series  $\eta(TV^{[u]}, t)$  of  $TV^{[u]}$  is a series in the variable  $t^{|u|}$ .

**Theorem 2.23.** *One has*

$$T^+V \simeq \otimes_{u \in L}^> T^+V^{[u]}.$$

*More precisely, the map  $\mu : \otimes_{u \in L}^> T^+V^{[u]} \rightarrow T^+V$ , which is the multiplication map in each summand, is an isomorphism in the category  ${}^{H_0}_{H_0}\mathcal{YD}$ . In particular, the Hilbert series of  $TV$  is  $\eta(TV, t) = \prod_{u \in L} \eta(TV^{[u]}, t)$ .*

*Proof.* This is a reformulation of Corollary 2.19.  $\square$

### 3. HOPF ALGEBRAS GENERATED BY A HOPF SUBALGEBRA AND A VECTOR SPACE

Our aim is now to apply the results of the previous section to arbitrary Hopf algebras. In many (though not all) cases we will be able to provide some new structure results.

Let  $H$  be a Hopf algebra with a filtration

$$0 = \mathcal{F}_{-1} \subset \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subseteq H.$$

The filtration is a *Hopf algebra filtration* if

- (1) It is a filtration:  $H = \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$ ,
- (2) it is an algebra filtration:  $\mathcal{F}_n \mathcal{F}_m \subseteq \mathcal{F}_{n+m}$ ,
- (3) it is a coalgebra filtration:  $\Delta(\mathcal{F}_n) \subseteq \sum_{i+j=n} \mathcal{F}_i \otimes \mathcal{F}_j$ , and
- (4) it behaves well with the antipode:  $S(\mathcal{F}_n) \subseteq \mathcal{F}_n$ .

Notice that in this case  $\mathcal{F}_0$  is a Hopf subalgebra of  $H$ . We will consider Hopf algebras with a Hopf algebra filtration satisfying a stronger version of condition (2):

**Assumption 3.1.**  $\mathcal{F}_n \mathcal{F}_m = \mathcal{F}_{n+m}$ .

A Hopf algebra with a Hopf algebra filtration satisfying Assumption 3.1 can be presented in the following way.

Suppose that the Hopf algebra  $H$  is generated (as an algebra) by a Hopf subalgebra  $H_0$  and a vector space  $V$ , such that

$$(3.2) \quad \Delta V \subseteq V \otimes H_0 + H_0 \otimes V + H_0 \otimes H_0, \text{ and}$$

$$(3.3) \quad S(V) \subseteq KVK.$$

We define  $\mathcal{F}_0 = H_0$ ,  $\mathcal{F}_1 = H_0 + H_0 V H_0$ , and  $\mathcal{F}_n = (\mathcal{F}_1)^n$ . Then  $\mathcal{F}_*$  is a Hopf algebra filtration which satisfies Assumption 3.1.

As an important example, assume that  $H$  is generated by grouplikes and skew-primitive elements and take  $H_0$  to be the subalgebra generated by the grouplikes and  $V$  to be the subspace generated by skew-primitives.

For  $n \geq 0$  let  $H_n = \mathcal{F}_n / \mathcal{F}_{n-1}$ . Let then

$$\text{gr}_{\mathcal{F}} H = \bigoplus_{n \geq 0} \mathcal{F}_n / \mathcal{F}_{n-1} = \bigoplus_{n \geq 0} H_n$$

be the associated graded Hopf algebra. Then we can consider the projection  $\pi : \text{gr}_{\mathcal{F}} H \rightarrow H_0$  and the inclusion  $\iota : H_0 \rightarrow \text{gr}_{\mathcal{F}} H$ , and this allows to write  $\text{gr}_{\mathcal{F}} H = R \# H_0$  as the smash product of  $H_0$  and the right coinvariants  $R = \{x \in \text{gr}_{\mathcal{F}} H \mid (\text{id} \otimes \pi) \Delta(x) = x \otimes 1\}$ . This is now a standard procedure: the last part is due to Radford [Rad85] and Majid [Maj94], while the first part is a modification of the one due to Andruskiewitsch and Schneider [AS98].

Also,  $R$  is a braided Hopf algebra in  ${}_{H_0}^{H_0} \mathcal{YD}$ , and  $R = \bigoplus_{n \geq 0} R_n$ ,  $R_0$  is the base field and  $R_1$  is a Yetter–Drinfel’d module which generates  $R$  because of Assumption 3.1. We let  $V := R_1$  and we make the following assumption:

**Assumption 3.4.**  $V = \bigoplus_{i=1}^d V_i$  is a direct sum of Yetter–Drinfel’d modules over  $H_0$ .

*Remark 3.5.* If  $V$  is irreducible, the methods in this paper do not yield any information on  $R$ . Otherwise,  $V$  has a nontrivial flag  $V = V_1 \supset V_2 \supset \dots \supset V_d$  of Yetter–Drinfel’d submodules over  $H_0$ . Consider on the vector space  $R_n$  the  $\mathbb{Z}$ -filtration  $R_n = R_{n,n} \supset R_{n,n+1} \supset \dots$ , where  $R_{n,m} = \sum_{i_1+\dots+i_n \geq m} V_{i_1} V_{i_2} \dots V_{i_n}$ . Then

$$R' := \bigoplus_{n=0}^{\infty} R'_n, \quad \text{where } R'_n = \bigoplus_{m \geq n} R_{n,m} / R_{n,m+1},$$

is a graded braided Hopf algebra in the category  ${}_{H_0}^{\text{H}_0}\mathcal{YD}$ , and all of the following considerations may be applied to  $R'$  instead of  $R$ .

Consider the projection  $TV \rightarrow R$ . We want to study the images under this projection of the components  $T^+V^{[u]}$  appearing in Theorem 2.23. We begin by considering the comultiplication in  $TV \# H_0$ .

We write the left  $H_0$ -coaction on  $r \in R$  by  $\delta(r) = r_{(-1)} \otimes r_{(0)}$ . Recall that the coproduct in the smash product  $R \# H_0$  is given by  $\Delta(r \# h) = (r^{(1)} \# r^{(2)}_{(-1)} h_{(1)}) \otimes (r^{(2)}_{(0)} \# h_{(2)})$ , where  $\Delta_R(r) =: r^{(1)} \otimes r^{(2)}$  is the coproduct of the braided Hopf algebra  $R$ . This notation applies in particular for  $R = TV$ .

**Proposition 3.6.** *Let  $X \in TV \subseteq TV \# H_0$  be a  $u$ -[ ]-letter. Then the coproduct  $\Delta$  of  $TV \# H_0$  satisfies*

$$\Delta(X) = X \otimes 1 + X_{(-1)} \otimes X_{(0)} + \sum_i (X'_i h_i) \otimes X''_i,$$

where  $X'_i, X''_i \in T^+V$ ,  $h_i \in H_0$ , and each  $X'_i$  is a  $w_i$ -[ ]-word with  $w_i \in M_{>u}$ .

*Proof.* We proceed by induction on  $|u|$ . If  $u \in A$ , we get  $X \in V$  and then  $\Delta(X) = X \otimes 1 + X_{(-1)} \otimes X_{(0)}$ , whence we are done. Assume now that  $|u| \geq 2$ ,  $\text{III}(u) = (v, w)$  and  $x$  is a  $u$ -vector. Then  $x = x_v x_w$ , and we write  $X = [x]$ ,  $Y = [x_v]$ ,  $Z = [x_w]$ . By standard computations,

$$\begin{aligned} \Delta(X) &= \Delta([Y, Z]) = \Delta(YZ - Z_{(0)}(S^{-1}(Z_{(-1)}).Y)) \\ &= (Y^{(1)}Y^{(2)}_{(-1)} \otimes Y^{(2)}_{(0)}) \cdot (Z^{(1)}Z^{(2)}_{(-1)} \otimes Z^{(2)}_{(0)}) \\ (3.7) \quad &\quad - (Z_{(0)}^{(1)}Z_{(0)}^{(2)}_{(-1)} \otimes Z_{(0)}^{(2)}_{(0)}) \\ &\quad \cdot ((S^{-1}(Z_{(-1)}).Y^{(1)})S^{-1}(Z_{(-2)}).Y^{(2)}_{(-1)}Z_{(-4)} \\ &\quad \quad \quad \otimes S^{-1}(Z_{(-3)}).Y^{(2)}_{(0)}). \end{aligned}$$

Note that  $Y$  is a  $v$ -[ ]-letter and  $Z$  is a  $w$ -[ ]-letter. According to the induction hypothesis and Lemma 2.12(1), for any  $h \in H_0$ ,  $h.Y^{(1)}$  can be

taken to be either  $\varepsilon(h)1$ ,  $h.Y$  or a  $v'[-]$ -word with  $v' \in M_{>v}$ . Similarly,  $h.Z^{(1)}$  can be taken to be either  $\varepsilon(h)1$ ,  $h.Z$  or a  $w'[-]$ -word with  $w' \in M_{>w}$ , and  $Z_{(0)}^{(1)}$  to be either  $1$ ,  $Z_{(0)}$  or a  $w'[-]$ -word with  $w' \in M_{>w}$ .

We begin by considering the summand

$$\begin{aligned} & (Y^{(1)}Y^{(2)}_{(-1)} \otimes Y^{(2)}_{(0)}) \cdot (Z^{(1)}Z^{(2)}_{(-1)} \otimes Z^{(2)}_{(0)}) \\ &= Y^{(1)}Y^{(2)}_{(-1)}Z^{(1)}Z^{(2)}_{(-1)} \otimes Y^{(2)}_{(0)}Z^{(2)}_{(0)} \\ &= Y^{(1)}(Y^{(2)}_{(-2)} \cdot Z^{(1)})Y^{(2)}_{(-1)}Z^{(2)}_{(-1)} \otimes Y^{(2)}_{(0)}Z^{(2)}_{(0)}. \end{aligned}$$

We consider the summands in which  $Y^{(1)}$  is a  $v'[-]$ -word with  $v' \in M_{>v}$ . Notice that since these  $v'$  are shorter than  $v$ , they belong to  $M_{>vw}$ . Therefore, since  $Z^{(1)}$  is either  $1$  or a  $t[-]$ -word with  $t \in M_{>vw}$ , by Lemma 2.16 these summands satisfy the claim of the Proposition. The summands in which  $Y^{(1)} = 1$  and  $Z^{(1)} \neq 1$  also satisfy the claim, because  $w > u$ . We are thus left with

$$(3.8) \quad Y_{(-1)}Z_{(-1)} \otimes Y_{(0)}Z_{(0)} + YZ^{(1)}Z^{(2)}_{(-1)} \otimes Z^{(2)}_{(0)}.$$

We consider now the other summand of  $\Delta(X)$ . By similar reasons, we are left with

$$\begin{aligned} (3.9) \quad & -Y_{(-1)}Z_{(-2)} \otimes Z_{(0)}(S^{-1}(Z_{(-1)}).Y_{(0)}) \\ & -Z_{(0)}^{(1)}Z_{(0)}^{(2)}_{(-1)}(S^{-1}(Z_{(-1)}).Y)S^{-1}(Z_{(-2)})Z_{(-4)} \\ & \quad \otimes Z_{(0)}^{(2)}_{(0)}(S^{-1}(Z_{(-3)}).1) \\ & = -Y_{(-1)}Z_{(-2)} \otimes Z_{(0)}(S^{-1}(Z_{(-1)}).Y_{(0)}) \\ & \quad -Z_{(0)}^{(1)}Z_{(0)}^{(2)}_{(-1)}(S^{-1}(Z_{(-1)}).Y) \otimes Z_{(0)}^{(2)}_{(0)} \\ & = -Y_{(-1)}Z_{(-2)} \otimes Z_{(0)}(S^{-1}(Z_{(-1)}).Y_{(0)}) \\ & \quad -Z_{(0)}^{(1)}((Z_{(-2)}^{(2)}S^{-1}(Z_{(-3)}^{(2)}).S^{-1}(Z_{(-1)}^{(1)})).Y)Z_{(-1)}^{(2)} \otimes Z_{(0)}^{(2)} \\ & = -Y_{(-1)}Z_{(-1)} \otimes mc^{-1}(Y_{(0)} \otimes Z_{(0)}) \\ & \quad -mc^{-1}(Y \otimes Z^{(1)})Z_{(-1)}^{(2)} \otimes Z_{(0)}^{(2)}. \end{aligned}$$

Adding (3.8) and (3.9) we get

$$Y_{(-1)}Z_{(-1)} \otimes [Y_{(0)}, Z_{(0)}] + [Y, Z^{(1)}]Z_{(-1)}^{(2)} \otimes Z_{(0)}^{(2)}.$$

Notice that  $Y_{(-1)}Z_{(-1)} \otimes [Y_{(0)}, Z_{(0)}] = X_{(-1)} \otimes X_{(0)}$ . Also, when in the second summand we put  $Z^{(1)} = Z$ , we get  $X \otimes 1$ . Finally, when  $Z^{(1)}$  is a  $w'[-]$ -word with  $w' \in M_{>w}$ , by using Lemmas 2.12(3), 2.14 and 2.16 we obtain terms which satisfy the claim.  $\square$

**Corollary 3.10.** *Let  $u \in L$  and  $v \in M_{\geq u}$ . Let  $X \in TV \subseteq TV \# H_0$  be a  $v$ - $[\cdot]$ -word. Then the coproduct  $\Delta$  of  $TV \# H_0$  satisfies*

$$\Delta(X) = X \otimes 1 + X_{(-1)} \otimes X_{(0)} + \sum_i (X'_i h_i) \otimes X''_i,$$

where  $X'_i, X''_i \in T^+V$ ,  $h_i \in H_0$ , and each  $X'_i$  is a  $w_i$ - $[\cdot]$ -word with  $w_i \in M_{\geq u}$ .

*Proof.* This follows at once from the Proposition and Lemma 2.16.  $\square$

We study the structure of  $R$  now. For that, we will use the following notation.

**Definition 3.11.** For  $u \in L$ , let  $\mathcal{V}^{[\geq u]}$  be the subalgebra of  $TV$  generated by  $(\sum_{v \in L, v \geq u} V^{[v]})$ . Let  $\mathcal{I}^{[\geq u]}$  be the ideal of  $\mathcal{V}^{[\geq u]}$  generated by  $(\sum_{v \in L, v > u} V^{[v]})$ . We define also  $V_{[u]} = \pi(V^{[u]})$ ,  $\mathcal{V}_{[\geq u]} = \pi(\mathcal{V}^{[\geq u]})$ ,  $\mathcal{V}_{[\geq u]}^+ = \pi(\mathcal{V}^{[\geq u]}) \cap \ker \varepsilon$ , and  $\mathcal{I}_{[\geq u]} = \pi(\mathcal{I}^{[\geq u]})$ , where  $\pi : TV \rightarrow R$  is the canonical projection.

The grading on  $R$  induces a grading on all of the algebras and ideals defined above. Take  $u \in L$ . Notice that the graded algebras  $\mathcal{V}^{[\geq u]} / \mathcal{I}^{[\geq u]}$  and  $\mathcal{V}_{[\geq u]} / \mathcal{I}_{[\geq u]}$  have only elements in degrees  $m|u|$  where  $m \in \mathbb{N}_0$ . Moreover, Lemma 2.16 and Corollary 2.21 imply that  $\mathcal{V}^{[\geq u]}$  is the subspace of  $TV$  generated by  $w$ - $[\cdot]$ -words with  $w \in M_{\geq u}$  and  $\mathcal{I}^{[\geq u]}$  is the subspace of  $\mathcal{V}^{[\geq u]}$  generated by  $w$ - $[\cdot]$ -words with  $w = w_1 \cdots w_n \in M_{\geq u}$  with  $w_1 > u$ .

Notice that the leading vector of any  $X \in \mathcal{V}^{[\geq u]} \setminus \mathcal{I}^{[\geq u]}$  of degree  $m|u|$  is a  $u^m$ -vector. Thus, we can choose for any  $u \in L$  a graded linear map  $\iota_u : \mathcal{V}_{[\geq u]}^+ / \mathcal{I}_{[\geq u]} \rightarrow T^+V^{[u]} = \bigoplus_{m \in \mathbb{N}} V^{[u^m]} \subset \mathcal{V}^{[\geq u]}$  such that  $\pi_u \circ \pi|_{T^+V^{[u]}} \circ \iota_u = \text{id}$ , where  $\pi : TV \rightarrow R$  and  $\pi_u : \mathcal{V}_{[\geq u]} \rightarrow \mathcal{V}_{[\geq u]} / \mathcal{I}_{[\geq u]}$  are the canonical maps.

Recall Def. 2.22 for the definition of  $\otimes^>$ .

**Theorem 3.12.** *The map  $\phi : \otimes_{u \in L} \mathcal{V}_{[\geq u]}^+ / \mathcal{I}_{[\geq u]} \rightarrow R^+$  defined by*

$$(3.14) \quad \phi(X_{u_1} \otimes \cdots \otimes X_{u_n}) = \pi(\iota_{u_1}(X_{u_1}) \cdots \iota_{u_n}(X_{u_n})),$$

where  $u_1, \dots, u_n \in L$ ,  $u_1 > \cdots > u_n$ , and  $X_{u_i} \in \mathcal{V}_{[\geq u_i]}^+ / \mathcal{I}_{[\geq u_i]}$  for all  $i$ , is an isomorphism of graded vector spaces.

To prove the theorem we will use the following lemma.

**Lemma 3.15.** *Let  $u = u_1^{n_1} \cdots u_r^{n_r} \in M$ , and let  $X_i \in V^{[u_i^{n_i}]}$  for all  $i$ . Take  $Y_i = \pi_{u_i}(\pi(X_i)) \in \mathcal{V}_{[\geq u_i]}^+ / \mathcal{I}_{[\geq u_i]}$ . Then*

$$X_1 \cdots X_r - \iota_{u_1}(Y_1) \cdots \iota_{u_r}(Y_r) \in \ker \pi + \sum_{\substack{w > u \\ w \in M}} V^{[w]}.$$

*Proof.* Since  $\pi_{u_i}\pi(X_i - \iota_{u_i}(Y_i)) = 0$ , we have  $X_i - \iota_{u_i}(Y_i) \in \ker \pi + \mathcal{I}^{[\geq u_i]}$ . We then consider

$$\begin{aligned} Z &= X_1 \cdots X_r - \iota_{u_1}(Y_1) \cdots \iota_{u_r}(Y_r) \\ &= \sum_{j=1}^r X_1 \cdots X_{j-1}(X_j - \iota_{u_j}(Y_j))\iota_{u_{j+1}}(Y_{j+1}) \cdots \iota_{u_r}(Y_r) \\ &\in \sum_{j=1}^r V^{[u_1^{n_1}]} \cdots V^{[u_{j-1}^{n_{j-1}}]} \mathcal{I}^{[\geq u_j]} V^{[u_{j+1}^{n_{j+1}}]} \cdots V^{[u_r^{n_r}]} + \ker \pi. \end{aligned}$$

As mentioned above the theorem,  $\mathcal{I}^{[\geq u_j]}$  consists of sums of  $w$ -[]-words where  $w$  runs on monotonic super-words  $> u_j^{n_j}$ . Thus, by Corollary 2.21 and Lemma 2.6,  $Z$  is a sum of  $w$ -[]-words, where  $w$  runs on super-words  $> u$ .  $\square$

*Proof of the Theorem.* Since the set of words of a given length is finite, the surjectivity of  $\phi$  follows easily from the previous lemma.

We now prove injectivity of  $\phi$ . To do so define  $\phi' : \otimes_{u \in L}^> \mathcal{V}_{[\geq u]}^+ / \mathcal{I}_{[\geq u]} \rightarrow T^+ V$  by

$$\phi'(X_{u_1} \otimes \cdots \otimes X_{u_n}) = \iota_{u_1}(X_{u_1}) \cdots \iota_{u_n}(X_{u_n}),$$

where  $u_1, \dots, u_n \in L$ ,  $u_1 > \cdots > u_n$ , and  $X_{u_i} \in \mathcal{V}_{[\geq u_i]}^+ / \mathcal{I}_{[\geq u_i]}$  for all  $i$ . Assume then that there exists a smallest integer  $m$  such that  $\phi$  is not injective in degree  $m$ . For all  $u \in L$  let  $B_u = \{b_{u,i} \mid i \in I_u\}$  be a homogeneous basis of  $\mathcal{V}_{[\geq u]}^+ / \mathcal{I}_{[\geq u]}$ , where  $I_u$  is an appropriate index set, and let  $X_{u,i} := \iota_u(b_{u,i})$  for all  $u \in L$ ,  $i \in I_u$ .

Suppose that there exists a nonempty finite subset  $M'$  of  $M$  with  $|w| = m$  for  $w \in M'$ , and for each  $w = w_1^{n_1} \cdots w_r^{n_r} \in M'$  there exist nonzero elements  $b_w \in \mathcal{V}_{[\geq w_1]}^+ / \mathcal{I}_{[\geq w_1]} \otimes \cdots \otimes \mathcal{V}_{[\geq w_r]}^+ / \mathcal{I}_{[\geq w_r]}$  such that  $\phi'(\sum_{w \in M'} b_w) \in \ker \pi$ . Let  $u = u_1^{m_1} \cdots u_s^{m_s}$ , where  $u_1 > \cdots > u_s$ , be the minimal element of  $M'$ , and write  $b_u := \sum_{i_1, \dots, i_s} \lambda_{i_1, \dots, i_s} b_{u_1, i_1} \otimes \cdots \otimes b_{u_s, i_s}$  with  $\lambda_{i_1, \dots, i_s} \in \mathbf{k}$ . We consider  $TV$  as a subalgebra of  $TV \# H_0$ , and then we have

$$\begin{aligned} \Delta(\phi'(\sum_{w \in M'} b_w)) &=: \sum_i Z'_i \otimes Z''_i \\ &\in (\ker \pi \# H_0) \otimes TV + (TV \# H_0) \otimes \ker \pi. \end{aligned}$$

Therefore,

$$(3.16) \quad \sum_i S^{-1}(Z''_{i(-1)}) Z'_i \otimes Z''_{i(0)} \in \ker \pi \otimes TV + TV \otimes \ker \pi.$$

We apply Prop. 3.6 to each  $[]$ -letter in  $\phi'(\sum_{w \in M'} b_w)$ , and we use Lemma 2.8 to obtain a description of the tensor factors of (3.16). Afterwards, we apply Corollary 2.21 and Lemma 2.6 to rearrange the tensor factors as sums of  $[]$ -words. This gives

$$\begin{aligned} & \sum_i S^{-1}(Z''_{i(-1)}) Z'_i \otimes Z''_{i(0)} \\ & \in \sum_{i_1, \dots, i_s} \lambda_{i_1, \dots, i_s} S^{-1}(X_{u_1, i_1(-1)}) X_{u_1, i_1(-2)} X_{u_2, i_2} \cdots X_{u_s, i_s} \otimes X_{u_1, i_1(0)} \\ & \quad + \sum_{\substack{w', w'' \in M \\ w' > u_2^{m_2} \cdots u_s^{m_s} \text{ or } w'' > u_1^{m_1}}} V^{[w']} \otimes V^{[w'']}. \end{aligned}$$

By repeatedly using Lemma 3.15 and since  $\phi'(\sum_{w \in M'} b_w) \in \ker \pi$ , we get

$$\begin{aligned} & \sum_i S^{-1}(Z''_{i(-1)}) Z'_i \otimes Z''_{i(0)} \\ & \in \sum_{i_1, \dots, i_s} \lambda_{i_1, \dots, i_s} \phi'(b_{u_2, i_2}) \cdots \phi'(b_{u_s, i_s}) \otimes \phi'(b_{u_1, i_1}) \\ & \quad + \sum_{\substack{w', w'' \in M, |w'|, |w''| < |u|, \\ w' > u_2^{m_2} \cdots u_s^{m_s} \text{ or } w'' > u_1^{m_1}}} (\text{Im } \phi' \cap V^{[w']}) \otimes (\text{Im } \phi' \cap V^{[w'']}) \\ & \quad + \ker \pi \otimes TV + TV \otimes \ker \pi. \end{aligned}$$

Therefore, (3.16) shows that

$$\begin{aligned} (3.17) \quad & \sum_{i_1, \dots, i_s} \lambda_{i_1, \dots, i_s} \phi'(b_{u_2, i_2} \cdots b_{u_s, i_s}) \otimes \phi'(b_{u_1, i_1}) \\ & \in \sum_{\substack{w', w'' \in M, |w'|, |w''| < |u|, \\ w' > u_2^{m_2} \cdots u_s^{m_s} \text{ or } w'' > u_1^{m_1}}} (\text{Im } \phi' \cap V^{[w']}) \otimes (\text{Im } \phi' \cap V^{[w'']}) \\ & \quad + \ker \pi \otimes TV + TV \otimes \ker \pi. \end{aligned}$$

By the assumption on  $m$ ,  $\pi \circ \phi' = \phi$  is injective in degrees  $< m$  and hence the sums

$$\begin{aligned} & (\ker \pi \cap V^{\otimes n}) + \left( \bigoplus_{\substack{w' \in M, |w'| = n, \\ w' > u_2^{m_2} \cdots u_s^{m_s}}} \text{Im } \phi' \cap V^{[w']} \right) + (\text{Im } \phi' \cap V^{[u_2^{m_2} \cdots u_s^{m_s}]}) \\ & (\ker \pi \cap V^{\otimes n}) + \left( \bigoplus_{\substack{w'' \in M, |w''| = n, \\ w'' > u_1^{m_1}}} \text{Im } \phi' \cap V^{[w'']} \right) + (\text{Im } \phi' \cap V^{[u_1^{m_1}]}) \end{aligned}$$

are direct in  $TV$  whenever  $1 \leq n < m$ . Thus (3.17) implies that  $\lambda_{i_1, \dots, i_s} = 0$  for all  $i_1, \dots, i_s$ , which contradicts to the choice of  $b_u$ .  $\square$

**Corollary 3.18.** *The Hilbert series of  $R$  factors as*

$$\eta(R, t) = \prod_{u \in L} \eta(\mathcal{V}_{[ \geq u]} / \mathcal{I}_{[ \geq u]}, t)$$

$\square$

The importance of Corollary 3.18 becomes clearer with the following theorem.

**Theorem 3.19.** *For each  $u \in L$ , the algebra  $\mathcal{V}_{[ \geq u]} / \mathcal{I}_{[ \geq u]} \# H_0$  is a  $|u| \mathbb{Z}$ -graded Hopf algebra, where the grading is induced by that of  $R \# H_0$ . Equivalently,  $\mathcal{V}_{[ \geq u]} / \mathcal{I}_{[ \geq u]}$  is a  $|u| \mathbb{Z}$ -graded braided Hopf algebra in  ${}_{H_0}^{\text{H}_0} \mathcal{YD}$ . Moreover,  $\mathcal{V}_{[ \geq u]} / \mathcal{I}_{[ \geq u]}$  is generated by  $V_{[u]} / (\mathcal{I}_{[ \geq u]} \cap V_{[u]})$  and it projects onto the Nichols algebra  $\mathcal{B}(V_{[u]} / (\mathcal{I}_{[ \geq u]} \cap V_{[u]}))$ . The quotient*

$$\eta\left(\mathcal{V}_{[ \geq u]} / \mathcal{I}_{[ \geq u]}, t\right) / \eta\left(\mathcal{B}(V_{[u]} / (\mathcal{I}_{[ \geq u]} \cap V_{[u]})), t^{|u|}\right)$$

*is a power series with nonnegative integer coefficients.*

*Proof.* Since  $\mathcal{V}_{[ \geq u]} / \mathcal{I}_{[ \geq u]}$  is graded and its degree 0 part is  $H_0$ , it is enough to prove that  $\mathcal{V}_{[ \geq u]} / \mathcal{I}_{[ \geq u]} \# H_0$  is a bialgebra (see [Tak71]). The Theorem follows from Proposition 1.1, by taking  $A = R \# H_0$ ,  $B = \mathcal{V}_{[ \geq u]}$ , and  $I = \mathcal{I}_{[ \geq u]}$ . It remains to show that Equations (1.2) hold in this case. Indeed, it is enough to prove this for generators of  $B$  and  $I$ , and  $[ ]$ -letters satisfy (1.2) thanks to Proposition 3.6.

By the definition of  $\mathcal{V}_{[ \geq u]}$  and  $\mathcal{I}_{[ \geq u]}$ ,  $\mathcal{V}_{[ \geq u]} / \mathcal{I}_{[ \geq u]}$  is generated as an algebra by the space  $V_{[u]} / (\mathcal{I}_{[ \geq u]} \cap V_{[u]})$ . Further,  $\mathcal{V}_{[ \geq u]} / \mathcal{I}_{[ \geq u]}$  can be considered as the quotient of  $T(V_{[u]} / (\mathcal{I}_{[ \geq u]} \cap V_{[u]}))$  by a graded Hopf ideal consisting of elements of degree  $\geq 2$ . Since  $\mathcal{B}(V_{[u]} / (\mathcal{I}_{[ \geq u]} \cap V_{[u]}))$  is the quotient of  $T(V_{[u]} / (\mathcal{I}_{[ \geq u]} \cap V_{[u]}))$  by the maximal Hopf ideal consisting of elements of degree  $\geq 2$ , there exists a natural projection  $\mathcal{V}_{[ \geq u]} / \mathcal{I}_{[ \geq u]} \rightarrow \mathcal{B}(V_{[u]} / (\mathcal{I}_{[ \geq u]} \cap V_{[u]}))$ . The last statement follows from Proposition 1.3.  $\square$

**Remark 3.20.** In Theorem 3.19 it is necessary to put  $t^{|u|}$  as the variable of the Hilbert series of the Nichols algebra, since in  $\mathcal{B}(V_{[u]} / (\mathcal{I}_{[ \geq u]} \cap V_{[u]}))$  the elements of  $V_{[u]} / (\mathcal{I}_{[ \geq u]} \cap V_{[u]})$  are considered to be in degree 1.

*Open Problems 3.21.*

- (1) Assume that  $R$  is a Nichols algebra. Are the graded Hopf algebras  $\mathcal{V}_{[ \geq u]} / \mathcal{I}_{[ \geq u]}$  appearing in Theorem 3.19 again Nichols algebras? This is true in the case where  $\text{char } \mathbf{k} = 0$ ,  $H_0$  is the group algebra of an abelian group, and  $R$  is finite dimensional, by

Kharchenko's PBW theorem. More generally, if  $R$  has a finite number of PBW generators, the statement follows by using the Weyl groupoid.

- (2) Generalize Theorems 3.12 and 3.19 to a more general setting which covers also Ufer's PBW basis.
- (3) Is it possible to generalize results of this paper to arbitrary (say finite dimensional) non-semisimple Hopf algebras?

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MATÍAS GRAÑA, DEPTO DE MATEMÁTICA - FCEyN, UNIVERSIDAD DE BUENOS AIRES, PAB I - CIUDAD UNIVERSITARIA (1428) BUENOS AIRES - ARGENTINA  
*E-mail address:* matiasg@dm.uba.ar

ISTVÁN HECKENBERGER, MATHEMATISCHES INSTITUT, UNIVERSITÄT LEIPZIG, AUGUSTUSPLATZ 10-11, 04109 LEIPZIG, GERMANY.  
*E-mail address:* heckenberger@math.uni-leipzig.de